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A BRIEF ACCOUNT OF H. GRASSMANN'S GEOMETRICAL THEORIES.*

By MR. ALEXANDER ŻIWET, Washington, D. C.

I. The fundamental relations with respect to which objects are considered in mathematics are the relations of equality and inequality, of variability and constancy.

If the objects under consideration are supposed to be *invariable*, two cases may be distinguished, corresponding respectively to the two branches of mathematics known as arithmetic and combinatory analysis.

1. We may regard the primary objects or elements as equal to one another; they are then called *units*, and by bringing these units into aggregates we form *numbers*.

2. We may regard the elements as differing from one another; aggregates of such elements are called *combinations*.

Thus, the units used in building up a number lose their individuality in the process, while in a combination the constituent elements remain distinct. The same units can form but one number, while from the same elements a variety of combinations may be derived. It thus becomes a fundamental law of arithmetic

*The principal mathematical works of Hermann Grassmann are:

1. *Die lineale Ausdehnungslehre, etc.* Leipzig, 1844. Reprinted in 1878 without material changes.

2. *Geometrische Analyse, etc.* Leipzig, 1847. A prize-essay of the Jablonowski Society.

3. *Die Ausdehnungslehre vollstaendig und in strenger Form bearbeitet.* Berlin, 1862.

4. Numerous memoirs on special subjects, published between 1842 and 1877 in *Crelle's Journal*, in the *Math. Annalen*, in Grunert's *Archiv*, and in the Goettingen *Abhandlungen*. One of these papers, "On the Essential Features of Grassmann's Extensive Algebra" (Grunert's *Archiv*, Vol. VI (1845), pp. 337–350) was translated into English by Professor W. W. Beman, and appeared in the *Analyst*, Vol. VIII (1881), pp. 96–7 and 114–124.

It is the object of the present paper to give, in the simplest form possible, a succinct account of Grassmann's mathematical theories and methods in their application to plane geometry. In the main, I have followed the presentation of the subject given by Victor Schlegel in his *System der Raumlehre* (2 Parts, Leipzig, 1872 and 1875), to which the student is referred for a fuller account than can here be attempted of a system of geometry based on Grassmann's methods.

that the order of aggregation, i. e. the arrangement of the elements in an aggregate, is immaterial. This commutative principle does not apply to the theory of combinations where the order of the elements is a distinctive feature of every form.

II. In considering a *variable* object, in its successive states of variation, we may in a similar manner apply the notion of equality and inequality.

1. In analysis, the various states of the primary object, here called *variable quantity*, are regarded as equal and are accordingly designated by the same symbol. Any form the relation of which to this variable remains unchanged while the latter varies, is a *function*.

2. In the science of extension (which as applied to space bears the name of geometry), the various states of the element, say for instance, the various positions of a moving point, are considered as different and are distinctly and individually exhibited in the resulting form, for instance in the line generated by the moving point.

III. It is this distinction between the analytical and geometrical points of view which Grassmann seeks to emphasize by giving the name *intensive* magnitude to the variable quantity or function of analysis, and *extensive* magnitude to the variable object of geometry, or, more generally, of the "science of extension" (*Ausdehnungslehre*).

It is characteristic of the intensive magnitude that although its various states are indicated by implication through the symbol that represents it, and may be derived from it, yet the individuality of these states is obliterated by the uniformity of the symbol. In the extensive magnitude, on the other hand, all the states of variation remain distinct and different one from another; they are, so to speak, crystallized in the resulting object.

Of course these distinctions are not absolute. By gradual steps we may effect the transition from number which originally is the representative of discrete quantity to the continuous quantity of what George Peacock has called "symbolical algebra." The same physical quantity may be regarded as an intensive as well as an extensive magnitude. Number, as Pascal has remarked, seems to imitate space though in itself of so widely different a nature. And, on the other hand, it has been pointed out by W. S. Jevons that the formation of the idea of number is not possible without the notion of diversity.

IV. In the science of extension, then, we consider a variable object and regard its successive states of variation as differing from one another, and as exhibited distinctly and individually in the resulting form.

Geometry, in Grassmann's view, is not a branch of pure mathematics, since it presupposes the idea of space, which does not originate in our mind but presents itself as something in nature outside and independent of the thinking mind.

Geometry, therefore, requires certain axioms, i. e. propositions describing the nature of space. The nature of the necessary axioms of geometry which has lately awakened such wide interest and given rise to an extensive branch of mathematical literature, was discussed by Grassmann as early as 1844 in so satisfactory a manner that the most recent authors on the subject* have found hardly anything to add to Grassmann's results. Grassmann was one of the first to clearly establish the idea of n-dimensional space; indeed, his science of extension may be regarded as a complete system of non-Euclidean geometry. In what follows, however, I shall content myself with illustrating his theories through their application to ordinary geometry; indeed I shall hardly ever pass beyond the narrow bounds of "Flatland," the home of Sylvester's celebrated two-dimensional book-worm.

V. In applying the general notions of the science of extension to the relations of space, we have to substitute for the general idea of variation the more special idea of *motion*, this being the kind of variation to which all spatial changes can be reduced and by which they may be represented.

The simplest spatial relation is position; its representative being the geometrical point. This, then, is the primary object, or element, which being subjected to spatial variation, i. e. motion, will produce geometrical objects of a more complicated type.

In saying, then, that a moving point generates a line, a moving line a surface a moving surface space, we mean that all the various states of the generator are combined in thought into one idea, that of the object generated, without, however, effacing the individuality of the elements. It follows that a surface, for instance, contains not only all the lines representing the various states passed through by the generating line, but at the same time all the points in every such line, these points representing the various states of the motion generating the line.

VI. The character of a geometrical object is, therefore, determined by two circumstances: 1. the nature of the generator; 2. the nature of the motion involved in the generation.

The simplest generator is the point; it has only one property: position. By subjecting it to motion we endow it with a new property: extension. Hence the resulting line has position and the extension generated by a single kind of motion, i. e. it has one dimension. A line in moving acquires an additional extension and becomes a surface characterized by the possession of position and two dimensions. Proceeding in the same way, we find space with three dimensions. Further we cannot go in ordinary geometry where we are confined to the con-

*See, for instance, Erdmann, *Die Axiome der Geometrie*, Leipzig, 1877; and A. Donadt, *Das mathematische Raumproblem und die geometrischen Axiome*, Leipzig, 1881.

sideration of the properties of "flat" or "homaloidal" space. But it is obvious that the general science of extension is not subject to any such limitations.

VII. Let us now see what are the principal characteristics of motion, and how these affect the generated objects.

Motion may be limited or unlimited, and accordingly the objects generated through motion may be unbounded, i. e. infinite, partially bounded, or completely bounded. Bounded objects, whether completely or partially, are said to have *magnitude;* the limits of the generating motion determine this magnitude.

The motion of any geometrical object is characterized by its direction; and the law according to which this direction changes from one state of the moving object to another state determines the *form* or *shape* of the generated object.

If the direction of motion remains throughout the same, the motion is called *simple*. The totality of all the objects that can be generated by a simple motion in a given direction (and its opposite) constitutes what Grassmann calls a *system* or field (*Gebiet*). Thus a straight line represents a system of the first *rank* or species (*Stufe*), as it contains all the points and segments that can be generated by a point moving in a given direction. If all the elements (points) of such a system of the first rank be subjected to a simple motion in a direction not contained in it, a system of the second rank, or a plane system, will be generated. Space thus appears to be a system of the third rank, since all its elements and all the objects contained in it may be generated by the simple motion of a system of the second rank.

The generalization of these relations led Grassmann to some very remarkable propositions connecting the numbers expressing the rank of extensive magnitudes and those expressing the rank of the systems in which they are contained and which they have in common. Dr. H. Noth, in his *Arithmetik der Lage* (Leipzig, 1882), has taken these relations as the starting-point of a new, purely geometrical calculus of position.

VIII. In order to treat geometrical objects analytically, not through the indirect method of co-ordinates, but directly, it became necessary to investigate the fundamental laws of algebraical operations, to determine how far these may without change be applied to extensive magnitudes and what modifications are required to make such application possible. Grassmann's investigations of the fundamental laws of algebra which closely agree with the results reached by Sir William R. Hamilton are now accepted by all modern authors on this subject. It may suffice to refer to H. Hankel's treatise on complex numbers* and to Dr. E. Schroeder's Algebra.†

THE SYSTEM OF THE POINT.

IX. The geometrical characteristic of a point is its position. A moving

* *Theorie der complexen Zahlensysteme*, etc.; Leipzig, 1867.

† *Lehrbuch der Arithmetik und Algebra*; Leipzig, 1873.

point changes its position; and two fixed points are geometrically distinguished by their different positions. Two or more points having the same position are therefore geometrically identical and must be designated by the same symbol, say e. All points having this position are said to belong to the system of this point.

The points of a system being all objects of the same kind may be combined into groups by simple numeration. The aggregate

$$\overset{(1)}{e} + \overset{(2)}{e} + \ldots + \overset{(a)}{e} = ae = a$$

is called a point-magnitude, or a *multiple point*. By means of the usual generalizations of algebra, we may extend this definition so as to embrace cases where a no longer designates a positive integer, but any real quantity, positive or negative, integer or fractional, rational or irrational, in short, a *scalar*.

We may now say that the system of the point e contains all the multiple points expressed by the symbol

$$a = ae,$$

where a may have any scalar value. The factor e, which may be called the geometrical unit of the system, is an extensive magnitude of zero dimensions; hence also the multiple point a. The scalar coefficient a transforms the purely geometrical object e into a quantity which allows of algebraical treatment. This quantity ae is said to be of the first degree since it contains the unit e of the system only once.

Any point of the system may be derived from any other point of the same system by multiplication with a scalar coefficient. Indeed, let $a = ae$, $b = \beta e$, be two points of the system e; we evidently have

$$a = \frac{a}{\beta} b, \qquad b = \frac{\beta}{a} a.$$

This may also be expressed by saying that

$$\beta a = ab, \quad \text{or} \quad \begin{vmatrix} a & a \\ b & \beta \end{vmatrix} = 0;$$

i. e. any two points of the same system are connected by a homogeneous linear equation.

The system of the point considered geometrically is of zero dimensions, while analytically it is a system of the first degree. In the terminology of Grassmann, who carefully avoids the term "dimension" as conveying the idea of space, which, in his theory, is merely a special form of extension, the "rank" of the system of the point is said to be zero or one according to the point of view from which it is considered.

In the geometrical applications of the science of extension, with which alone I am concerned here, the use of the terms "dimension" and "degree" as defined above cannot be deemed objectionable; it will, on the other hand, prevent any ambiguity likely to arise from the use of the word "rank" in two different meanings.

THE SYSTEM OF THE STRAIGHT LINE.

X. The straight line is generated by the *simple* motion of a point. It is, therefore, the simplest geometrical object of one dimension.

From the generating point the line derives its *position*, so that all lines having a point in common may be said to have the same position.

From the generating motion the line receives its second characteristic, viz. its *direction*. The idea of one direction, however, necessarily involves the idea of the opposite of this direction. A straight line, therefore, represents two opposite directions.

One point of a straight line determines its position. Two points are necessary and sufficient to completely determine the line, that is, to indicate its position and its two opposite directions.

To distinguish between these two directions we shall make use of the order of the determining points, thus indicating by AB a motion from the point A to the point B, and by BA a motion from B to A.

XI. The two points A and B determine not only the direction, but also the amount of the simple motion from one point to the other. It seems proper to designate this magnitude as a difference $B - A$ since it expresses the difference in position of the two points, and, like an algebraical difference, changes its sign if the order of its elements is reversed.

Although derived from somewhat different considerations, this magnitude is, in all essential respects, equivalent to Hamilton's *vector*, and I shall adopt this name.

We have then for the vector the expression

$$a = B - A = -(A - B),$$

which evidently may be considered an abbreviation of the ordinary representation of a segment of a straight line in the form of a difference of two abscissæ counted from an origin in the same line, thus:—

$$a = x_b - x_a = B - A.$$

It follows at once that we can operate with such vectors just as if they were ordinary algebraical differences. We have, for instance, for any number of points in a straight line

$$(B - A) + (C - B) + \ldots + (N - M) = (N - A),$$

or $$(B - A) + (C - B) + \ldots + (A - N) = 0,$$

where the parentheses may be omitted.

If n points $A,B, \ldots N$ be given, we can always find one, and only one, point M such as to satisfy the condition

$$(A - M) + (B - M) + \ldots + (N - M) = 0;$$

whence $$n \,.\, M = A + B + \ldots + N,$$

or $$M = \frac{1}{n}(A + B + \ldots + N).$$

This point M is called the *mean point* or *centre* of the n given points.

The expression $n \,.\, M = A + B + \ldots + N$, or, in a more systematic notation,

$$ae = e_1 + e_2 + \ldots + e_n$$

leads again to the consideration of *multiple points*, i. e. points affected with co-efficients.

XII. If two points of a straight line are given, any other point of this line may be expressed as a homogeneous linear function of the given points.

Indeed, let e_1 and e_2 be the given points, and e the point to be expressed; we can always assign three scalars a, a_1, a_2, making

$$ae = a_1e_1 + a_2e_2, \text{ where } a = a_1 + a_2.$$

$\frac{a_2}{a_1}$ is evidently the ratio in which the point e divides the distance between e_1 and e_2.

Two points, $a = a_1e_1 + a_2e_2$, $b = \beta_1e_1 + \beta_2e_2$, coincide if

$$\frac{a_1}{\beta_1} = \frac{a_2}{\beta_2}, \quad \text{or} \quad \begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0.$$

XIII. As we have seen, the expression

$$a = ae = a_1e_1 + a_2e_2, \text{ where } a = a_1 + a_2,$$

in general represents a point. If, however, the coefficient a of e vanishes, that is, if we have $a = a_1 + a_2 = 0$, the expression reduces to

$$a = a_1(e_1 - e_2) = a_2(e_2 - e_1),$$

and represents a multiple vector (see Art. XI). The ratio $a_2 : a_1$ becoming in this case $= -1$, we see that the vector is in a certain sense equivalent to the point at infinity of the line.

On the other hand, it is easily seen that any vector in a given line may be represensed in the form $a = a_1e_1 + a_2e_2$, where $a_1 + a_2 = 0$. Hence a homogeneous linear function of two points represents a point or a vector according as

the sum of the co-efficients of the two points is different from or equal to zero.

Points and vectors are, therefore, called *magnitudes of the first degree.*

It is characteristic of the system of the straight line that in it any magnitude of the first degree may be derived from any two such magnitudes. In other words, *any three magnitudes of the first degree in the same system are connected by a homogeneous linear equation.*

Indeed, let $a = \alpha_1 e_1 + \alpha_2 e_2$, $b = \beta_1 e_1 + \beta_2 e_2$, $c = \gamma_1 e_1 + \gamma_2 e_2$ be the three magnitudes. Eliminating e_1, e_2 between these equations we obtain the relation

$$\begin{vmatrix} a & \alpha_1 & \alpha_2 \\ b & \beta_1 & \beta_2 \\ c & \gamma_1 & \gamma_2 \end{vmatrix} = 0,$$

which is linear and homogeneous in a, b, c. Any one of these magnitudes may, therefore, be expressed by the two others, i. e. the line is a *system of the second degree.*

XIV. The *product* of two magnitudes of the first degree,

$$a = \alpha_1 e_1 + \alpha_2 e_2, \qquad b = \beta_1 e_1 + \beta_2 e_2,$$

is

$$ab = \alpha_1\beta_1 \cdot e_1 e_1 + \alpha_1\beta_2 \cdot e_1 e_2 + \alpha_2\beta_1 \cdot e_2 e_1 + \alpha_2\beta_2 \cdot e_2 e_2,$$

and its signification evidently depends upon the interpretation to be given to the products of simple points, such as $e_1 e_1$, $e_1 e_2$, etc.

These binary products of unit-points are to be regarded as units of the next higher degree. In Grassmann's system, the product $e_1 e_2$ is interpreted as representing a segment of the straight line determined by the points e_1 and e_2, equal in length and direction to the distance between these points.

This *linear segment* $e_1 e_2$, as I shall call it, differs from the vector $e_2 - e_1$ only in being confined to the line determined by e_1 and e_2; it is a localized vector, like Clifford's *rotor*, or *spin*. While two vectors are regarded as equal, whenever they are of equal length and direction, whether situated in the same line or in parallel lines, two segments will be called equal only if they are parts of the same line. The segment thus appears to be the true geometrical representative of a force in mechanics.

If the points e_1 and e_2 coincide, the segment vanishes; if they are interchanged, the sign of the segment is reversed. Hence the fundamental laws to which the symbol $e_1 e_2$ is subject:—

$$e_1 e_1 = 0, \qquad e_1 e_2 = -e_2 e_1.$$

Such, then, are the equations of condition for the kind of multiplication expressed by $e_1 e_2$. Here, as in the theory of quaternions, the commutative principle of algebraical or scalar multiplication is replaced by a law similar to that in the theory of determinants, where also the interchange of two contiguous ele-

ments reverses the sign of the result; on account of which characteristic this kind of multiplication has been called by R. F. Scott* *polar* multiplication. Professor J. Willard Gibbs, in his *Vector Analysis*, New Haven, 1881–4, uses the name *skew* multiplication, while Grassmann himself called it *outer* or *external* multiplication, giving as his reason that its products differ from zero only when the factors lie one outside of the other.

XV. If we now apply the above fundamental laws of external multiplication to the expression of the product ab as stated at the beginning of the preceding article, we obtain

$$ab = (a_1\beta_2 - a_2\beta_1)\, e_1e_2 = \begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_1e_2.$$

It is readily seen that

$$ba = (\beta_1a_2 - \beta_2a_1)e_1e_2 = \begin{vmatrix} \beta_1 & \beta_2 \\ a_1 & a_2 \end{vmatrix} e_1e_2 = -\, ab,$$

$$aa = a^2 = \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} e_1e_2 = 0\,;$$

i. e. *the fundamental laws of external multiplication apply not only to unit-points but to magnitudes of the first degree in general.*

Either one of the two laws

$$ab = -\, ba, \quad aa = 0,$$

may easily be shown to be a consequence of the other.

XVI. The product of any two magnitudes of the first degree,

$$ab = \begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_1e_2,$$

always represents a segment.

If a and b are both points, the segment ab vanishes only in the case

$$\begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0, \quad \text{or} \quad a_1 : a_2 = \beta_1 : \beta_2,$$

that is, when the points coincide.

If a and b are both vectors we have $a_1 + a_2 = 0$, $\beta_1 + \beta_2 = 0$; hence

$$ab = \begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_1e_2 = \begin{vmatrix} a_1 + a_2 & a_2 \\ \beta_1 + \beta_2 & \beta_2 \end{vmatrix} e_1e_2 = \begin{vmatrix} 0 & a_2 \\ 0 & \beta_2 \end{vmatrix} e_1e_2 = 0\,;$$

i. e. the product of two vectors in the same line vanishes. If a is a point and b a vector, we have $\beta_1 + \beta_2 = 0$; hence

*See his *Treatise on Determinants*, London, Cambridge, and Leipzig, 1880.

$$ab = \begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_1e_2 = \begin{vmatrix} a_1 + a_2 & a_2 \\ 0 & \beta_2 \end{vmatrix} e_1e_2 = (a_1 + a_2)\,\beta_2 e_1e_2.$$

The special cases

$$(e_1 - e_2)\,e_1 = e_1e_2, \quad (e_1 - e_2)\,e_2 = e_1e_2$$

show that a segment may always be regarded as the product of a vector of equal length and direction into either of the end-points of the segment.

XVII. Putting now $e_1e_2 = 1$, i. e. assuming a certain segment as unit of measure, we obtain a scalar expression for any segment of the system determined by e_1 and e_2.

In an ordinary algebraical product of the form $e_1e_2 = 1$, each factor is called the reciprocal of the other, and we write $e_1 = \frac{1}{e_2}$, or $e_1 = e_2^{-1}$. In the external product $e_1e_2 = 1$, the factors of which are not interchangeable, the second factor, which is always regarded as the multiplier, is called by Grassmann the *complement* (*Ergaenzung*) of the first, in symbols

$$e_2 = \mid e_1.$$

And since $e_1e_2 = 1 = -e_2e_1$, we have

$$-e_1 = \mid e_2;$$

whence, substituting for e_2 its value $\mid e_1$,

$$-e_1 = \mid e_2 = \parallel e_1.$$

Thus the double application of the sign $\mid$ reverses the algebraical sign of the quantity; in this respect, it has the same effect as the factor $i = \sqrt{-1}$.

Substituting $e_2 = \mid e_1$ in $e_1e_2 = 1$, and $-e_1 = \mid e_2$ in $-e_2e_1 = 1$, we find

$$e_1 \mid e_1 = 1, \quad e_2 \mid e_2 = 1;$$

multiplying $\mid e_2 = -e_1$ by e_1, and $\mid e_1 = e_2$ by e_2, we obtain

$$e_1 \mid e_2 = -e_1e_1 = 0, \quad e_2 \mid e_1 = e_2e_2 = 0.$$

The operation characterized by the laws

$$e_1 \mid e_2 = 0, \quad e_1 \mid e_1 = 1,$$

is called by Grassmann *inner* or *internal* multiplication, since here the value of the product is different from zero only when the factors coincide. Professor Gibbs uses the name *direct* multiplication.

To extend these definitions to magnitudes of the first degree in general, we define the complement, $\mid a$, of $a = a_1e_1 + a_2e_2$ by means of the equation

$$\mid a = a_1 \mid e_1 + a_2 \mid e_2 = a_1e_2 - a_2e_1.$$

If $a = a_1e_1 + a_2e_2$, $b = \beta_1e_1 + \beta_2e_2$ be any two magnitudes of the first degree,

we obtain as their internal product

$$\begin{aligned} a \mid b &= (a_1 e_1 + a_2 e_2)(\mathfrak{z}_1 e_2 - \mathfrak{z}_2 e_1) \\ &= (a_1 \mathfrak{z}_1 + a_2 \mathfrak{z}_2)\, e_1 e_2 \\ &= a_1 \mathfrak{z}_1 + a_2 \mathfrak{z}_2. \end{aligned}$$

Similarly we find for the internal square

$$a \mid a = a_1^2 + a_2^2.$$

[TO BE CONTINUED.]

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.

By MR. MARCUS BAKER, Washington, D. C.

[CONTINUED FROM VOL. I, PAGE 138.]

GROUP II. PART I.

$\Delta =$

32. $\frac{1}{4}\sqrt{8m_c^2(a^2+b^2-2m_c^2)-(a^2-b^2)^2}$

33. $\frac{1}{2}h_c(\sqrt{a^2-h_c^2}+\sqrt{b^2-h_c^2})$

34. $\frac{1}{2}\mathfrak{z}_c ab\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{1-\frac{1}{4}\mathfrak{z}_c^2\left(\frac{1}{a}+\frac{1}{b}\right)^2}$

35. $\frac{1}{3}\sqrt{4m_b^2m_c^2-[\frac{3}{4}a^2-(m_b^2+m_c^2)]^2}=\frac{2}{3}\sqrt{m_b^2m_c^2-k_a^4}$

where $2k_a^2=\frac{3}{4}a^2-(m_b^2+m_c^2)$, etc.

36. $\dfrac{h_b^3\sqrt{a^2-h_b^2}-h_c^3\sqrt{a^2-h_c^2}}{2(h_b^2-h_c^2)}$

37. $\frac{1}{6}h_a(\sqrt{4m_b^2-h_a^2}+\sqrt{4m_c^2-h_a^2})$

38. $\frac{1}{2}ah_a$.

39. $\frac{1}{2}\sqrt{abh_ah_b}=\frac{1}{2}\dfrac{a\pm b}{\frac{1}{h_a}\pm\frac{1}{h_b}}$

GROUP II. PART I.—*Continued.*

$\Delta =$

40.* $h_a \sqrt{\left(h_a^2 + \sqrt{(m_a^2 - h_a^2)(\beta_a^2 - h_a^2)}\right)\left(\sqrt{\frac{m_a^2 - h_a^2}{\beta_a^2 - h_a^2}} - 1\right)}$

41. $\frac{1}{2}h_a(\sqrt{\frac{1}{4}a^2 + m_a^2 - h_a^2 + a\sqrt{m_a^2 - h_a^2}} + \sqrt{\frac{1}{4}a^2 + m_a^2 - h^2 - a\sqrt{m_a^2 - h_a^2}})$

$= \frac{1}{2}h_a(\sqrt{\frac{1}{4}a^2 + k_a^2 + ak} + \sqrt{\frac{1}{4}a^2 + k_a^2 - ak})$, where $k_a = m_a^2 - h_a^2$, etc.

42. $\frac{Rh_bh_c}{a}$

43. $\frac{ab}{4R}\sqrt{2(a^2 + b^2 - 2m_c^2)}$

44. $h_a(\sqrt{b^2 - h_a^2} \pm \sqrt{m_a^2 - h_a^2})$

45.† $\frac{ab}{8R^2}(a\sqrt{4R^2 - b^2} + b\sqrt{4R^2 - a^2})$

46. $2R^2[(f - 2x^2)\sqrt{(g - 2x^2)(1 - g + 2x^2)}$

$+ (g - 2x^2)\sqrt{(f - 2x^2)(1 - f + 2x^2)}]$,

where $f = \frac{m_a^2 + 2m_b^2}{3R^2}$, $g = \frac{2m_a^2 + m_b^2}{3R^2}$, and $x\ (= \sin C)$ is to be found from the equation

$x = \sqrt{(f - 2x^2)(1 - g + 2x^2)} + \sqrt{(g - 2x^2)(1 - f + 2x^2)}$.

47. $\frac{sh_ar_a}{2(h_a + r_a)}$

*If μ and ϕ are auxiliaries determined from the equations $\cos\mu = \frac{h_a}{m_a}$ and $\cos\phi = \frac{h_a}{\beta_a}$, then

$$\Delta = \frac{h_a^2}{\cos\mu}\sqrt{\frac{\sin 2(\mu - \phi)}{\sin 2\phi}}.$$

[For this elegant expression I am indebted to my friend Mr. Charles H. Kummell, of the U. S. Coast and Geodetic Survey.]

†$\sqrt{R^2h_a^2 - \Delta^2} + \sqrt{R^2h_b^2 - \Delta^2} = \frac{1}{2}\left\{\frac{Rh_ah_b}{\Delta}\right\}^2$

GROUP II. PART II.

$\Delta =$

48. $\frac{1}{2}ab \sin C$

49. $\frac{1}{2}\frac{h_a h_b}{\sin C}$

50. $\frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} = \frac{\frac{1}{2}a^2}{\cot B + \cot C}$

51. $\frac{1}{2}h_a^2 \frac{\sin A}{\sin B \sin C} = \frac{1}{2}h_a^2 (\cot B + \cot C) = \frac{1}{2}h_a^2 (\sin 2B + \sin 2C)$

52. $\frac{1}{2}(a^2 - b^2) \frac{\sin A \sin B}{\sin (A - B)} = \frac{1}{4}(a^2 \sin 2B + b^2 \sin 2A) = \frac{\frac{1}{2}(a^2 - b^2)}{\cot A - \cot B}$

53. $\frac{1}{2}(h_b^2 - h_a^2) \frac{\sin A \sin B}{\sin (A - B)} \cdot \frac{1}{\sin^2 C} = \frac{1}{2}h_a h_b \frac{\tan \frac{1}{2}A \tan B \tan \frac{1}{2}C}{\cos A}$

$= h_a h_b \frac{\tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C}{\cos A (1 - \tan^2 \frac{1}{2}B)}$

54. $\frac{2m_a^2 \sin A \sin B \sin C}{2(\sin^2 A + \sin^2 B + \sin^2 C) - 3 \sin^2 A}$

55. $\frac{2(m_b^2 - m_a^2)}{3(\cot A - \cot B)}$

56. $\frac{1}{3}\sqrt{4m_b^2 m_c^2 - [9K^2 \sin^2 A - (m_b^2 + m_c^2)]^2}$

57. $\frac{2R^4}{m_a^2} \sin A \sin B \sin C (-\sin^2 A + 2 \sin^2 B + 2 \sin^2 C)$

58. $\frac{4m_a^2 + 3a^2}{8(\cot A + \cot B + \cot C)}$

59. $\frac{4(m_a^2 + m_b^2) - 3c^2}{4(\cot A + \cot B + \cot C)}$

60. $\frac{1}{2}\beta_a^2 \frac{\sin A}{\sin B \sin C} \sin (B + \frac{1}{2}A) \sin (A + \frac{1}{2}B)$

61. $Rh_a \sin A$

62. $\frac{1}{2}a\beta_a \sin (C + \frac{1}{2}A)$

63. $K^2 \sin 2A (1 + \cos C)$

64. $Ra \sin B \sin C = Ra (\cos A + \cos B \cos C)$

65. $Rh_a \tan A \sin B \tan \frac{1}{2}C$

66. $R\beta_a \tan A \sin B \tan \frac{1}{2}C \sin (C + \frac{1}{2}A)$

GROUP II. PART II.—*Continued.*

$\Delta =$

67. $\dfrac{h_a(h_b+h_c)}{4\sin(C+\frac{1}{2}A)\cos\frac{1}{2}A}$

68. $\dfrac{{}_i\beta_a(h_b+h_c)}{4\cos\frac{1}{2}A}$

69. $\frac{1}{4}{}_i\beta_a{}^2(b+c)\left(\dfrac{1}{b}+\dfrac{1}{c}\right)\tan\frac{1}{2}A.$

70. $\frac{1}{2}{}_i\beta_a{}_i\beta_b\dfrac{\sin(B+\frac{1}{2}A)\sin(A+\frac{1}{2}B)}{\sin C}$

71. ${}_i\beta_a{}_i\beta_b{}_i\beta_c\dfrac{-{}_i\beta_a\sin\frac{1}{2}A+{}_i\beta_b\sin\frac{1}{2}B+{}_i\beta_c\sin\frac{1}{2}C}{-{}_i\beta_b{}_i\beta_c\cos\frac{1}{2}A+{}_i\beta_a{}_i\beta_a\cos\frac{1}{2}B+{}_i\beta_a{}_i\beta_b\cos\frac{1}{2}C}$

72. $\frac{1}{2}a\sin B(a\cos B+\sqrt{b^2-a^2\sin^2 B})=\frac{1}{2}b\sin A(b\cos A+\sqrt{a^2-b^2\sin^2 A})$

GROUP III. PART I.

73. $sr=(s-a)r_a$

74. $\frac{1}{3}r(\sqrt{2m_a{}^2+2m_b{}^2-m_c{}^2}+\sqrt{2m_b{}^2+2m_c{}^2-m_a{}^2}+\sqrt{2m_c{}^2+2m_a{}^2-m_b{}^2})$
$=\frac{1}{3}r_a(\sqrt{2m_a{}^2+2m_b{}^2-m_c{}^2}-\sqrt{2m_b{}^2+2m_c{}^2-m_a{}^2}+\sqrt{2m_c{}^2+2m_a{}^2-m_b{}^2})$

75. $\sqrt{\dfrac{rr_br_c}{-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}}=\sqrt{\dfrac{r_ar_br_c}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}}$

76. $\dfrac{R}{2s}(h_ah_b+h_bh_c+h_ch_a)=\dfrac{R}{2(s-a)}(h_ah_b-h_bh_c+h_ch_a)$

77. $\sqrt{\frac{1}{2}Rr(h_ah_b+h_bh_c+h_ch_a)}=\sqrt{\frac{1}{2}Rr_a(h_ah_b-h_bh_c+h_ch_a)}$

78. $\dfrac{s}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=\dfrac{s-a}{-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}$

79. $\sqrt{\dfrac{r}{\left(-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}-\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}\right)}}$
$=\sqrt{\dfrac{r_a}{\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}-\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}\right)}}$

GROUP III. PART II.

$\Delta =$

80. $a(-\beta_a \sin \tfrac{1}{2}A + \beta_b \sin \tfrac{1}{2}B + \beta_c \sin \tfrac{1}{2}C)$
$= 2s(\beta_a \sin \tfrac{1}{2}A + \beta_b \sin \tfrac{1}{2}B + \beta_c \sin \tfrac{1}{2}C)$

81. $\sqrt{2r\beta_a\beta_b\beta_c\left[\dfrac{\cos(A-B)+\cos(B-C)+\cos(C-A)+1}{\cos A+\cos B+\cos C-1}\right]}$
$= \sqrt{2r_a\beta_a\beta_b\beta_c\left[\dfrac{\cos(A-B)+\cos(B-C)+\cos(C-A)+1}{-\cos A+\cos B+\cos C+1}\right]}$

82. $s^2 \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = (s-a)^2 \tan \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C$

83. $r^2 \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C = r_a^2 \cot \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C.$

GROUP IV. PART I.

84. $a\dfrac{rr_a}{r_a-r} = a\dfrac{r_br_c}{r_b+r_c}$

85. $rr_a\dfrac{r_b+r_c}{a} = r_br_c\dfrac{r_a-r}{a}$

86. $(a+b)\dfrac{rr_c}{r+r_c} = (a-b)\dfrac{r_ar_b}{r_a-r_b}$

87. $rr_a\dfrac{r_b-r_c}{b-c} = r_br_c\dfrac{r+r_a}{b+c}$

88. $rr_a\sqrt{\dfrac{r_b+r_c}{r_a-r}} = r_br_c\sqrt{\dfrac{r_a-r}{r_b+r_c}}$

89. $rr_a\sqrt{\dfrac{4R-(r_a-r)}{r_a-r}} = r_br_c\sqrt{\dfrac{4R-(r_b+r_c)}{r_b+r_c}}$

90. $\dfrac{sh_ar_a}{h_a+2r_a} = \dfrac{(s-a)h_ar}{h_a-2r}$

91. $r\sqrt{\dfrac{h_ar_br_c}{h_a-2r}} = r_a\sqrt{\dfrac{h_ar_br_c}{h_a+2r_a}}$

92. $\sqrt{\dfrac{rr_a}{\left(\dfrac{1}{h_a}-\dfrac{1}{h_b}+\dfrac{1}{h}\right)\left(\dfrac{1}{h_a}+\dfrac{1}{h_b}-\dfrac{1}{h_c}\right)}}$
$= \sqrt{\dfrac{r_br_c}{\left(\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}\right)\left(-\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}\right)}}$

GROUP IV. PART II.

$\Delta =$

93. $rr_a \cot \tfrac{1}{2}A = r_b r_c \tan \tfrac{1}{2}A$

GROUP V.

94. $r^2 \cot \tfrac{1}{2}A + 2Rr \sin A = r_a^2 \cot \tfrac{1}{2}A - 2Rr_a \sin A = r_b^2 \cot \tfrac{1}{2}A - 2Rr_b \sin A$
$= r_c^2 \cot \tfrac{1}{2}A - 2Rr_c \sin A.$

MISCELLANEOUS EXPRESSIONS FOR THE AREA OF A PLANE TRIANGLE.

If we designate the distances from the orthocentre to the sides of the triangle by k_a, k_b, k_c and from the orthocentre to the vertices by k_a', k_b', k_c', then

95. $\Delta = \tfrac{1}{2}(ak_a + bk_b + ck_c)$

96.* $\Delta = \tfrac{1}{4}(ak_a' + bk_b' + ck_c')$

97.* $\Delta = \dfrac{1}{8R}(a^2k_a' \operatorname{cosec} A + b^2k_b' \operatorname{cosec} B + c^2k_c' \operatorname{cosec} C).$

If we designate the distances between the centres of the escribed circles by d_a, d_b, d_c, then

98.* $\Delta = \tfrac{1}{4}\sqrt{2d_ad_bd_cr \sin A \sin B \sin C}$

99. $\Delta = \dfrac{d_ad_bd_cr}{16R^2}$

100. $\Delta = \dfrac{d_ad_bd_c}{4R} \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$

101.* $\Delta = r_ar_b\sqrt{\dfrac{d_c^2}{(r_a + r_b)^2} - 1}.$

If x_ay_a, x_by_b, x_cy_c designate the rectangular co-ordinates of a triangle in a plane, then

102. $\Delta = \tfrac{1}{2}[y_a(x_c - x_b) + y_b(x_a - x_c) + y_c(x_b - x_a)]$

and, in polar co-ordinates,

103. $\Delta = \tfrac{1}{2}[r_ar_b \sin(\theta'' - \theta') + r_br_c \sin(\theta''' - \theta'') + r_cr_a \sin(\theta' - \theta''')].$

If the equations of the sides of a triangle are

$$a_ax + b_ay + c_a = 0;\ a_bx + b_by + c_b = 0;\ a_cx + b_cy + c_c = 0,$$

then

104. $\Delta = \tfrac{1}{2}\left[\dfrac{[a_a(b_bc_c - b_cc_b) + a_b(b_cc_a - b_ac_c) + a_c(b_ac_b - b_bc_a)]^2}{(a_ab_b - b_aa_b)(a_bb_c - b_ba_c)(a_cb_a - b_ca_a)}\right].$

* *Tidsskrift for Mathematik*. 8°. Copenhagen, 1884, 5th Series, Vol. I, No. 2, pp. 78–79.
Mathesis. 8°. Gand, 1884, November, Vol. IV, p. 232.

If the co-ordinates of the vertices of a triangle are x_a, y_a, z_a; x_b, y_b, z_b; x_c, y_c, z_c, and

$$\begin{aligned} \alpha &= x_a^2 + y_a^2 + z_a^2, & \lambda &= x_b x_c + y_b y_c + z_b z_c, \\ \beta &= x_b^2 + y_b^2 + z_b^2, & \mu &= x_c x_a + y_c y_a + z_c z_a, \\ \gamma &= x_c^2 + y_c^2 + z_c^2; & \nu &= x_a x_b + y_a y_b + z_a z_b; \text{ then} \end{aligned}$$

105.* $$4\Delta^2 = -\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \alpha & \nu & \mu \\ 1 & \nu & \beta & \lambda \\ 1 & \mu & \lambda & \gamma \end{vmatrix}$$

or, without determinants,

$$\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4,$$

where
$$\begin{aligned} a^2 &= (x_b - x_c)^2 + (y_b - y_c)^2 + (z_b - z_c)^2, \\ b^2 &= (x_c - x_a)^2 + (y_c - y_a)^2 + (z_c - z_a)^2, \\ c^2 &= (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2, \end{aligned}$$

and if $\alpha, \beta, \gamma, \lambda, \mu$ and ν, have the values set down above, then

$$\begin{aligned} a^2 &= \beta + \gamma - 2\lambda, \\ b^2 &= \gamma + \alpha - 2\mu, \\ c^2 &= \alpha + \beta - 2\nu, \end{aligned}$$

and
$$\begin{aligned} \Delta^2 = {} & 2(\alpha + \beta - 2\nu)(\beta + \gamma - 2\lambda) - (\alpha + \beta - 2\nu)^2 \\ & + 2(\beta + \gamma - 2\lambda)(\gamma + \alpha - 2\mu) - (\beta + \gamma - 2\lambda)^2 \\ & + 2(\gamma + \alpha - 2\mu)(\alpha + \beta - 2\nu) - (\gamma + \alpha - 2\mu)^2, \end{aligned}$$

which reduces to

$$\Delta^2 = 4(\lambda + \mu + \nu)^2 - 8(a\lambda + \beta\mu + \gamma\nu) + 4(a\beta + \beta\gamma + \gamma a).$$

106. $\Delta = \frac{1}{8R}\left[(s-a)D_a^2 + (s-b)D_b^2 + (s-c)D_c^2 - sD^2\right]$ where D, D_a, D_b, D_c are the distances between the circumcentre and the in- and e-centres.

107. $\Delta = 4K$, where K is the area of the triangle formed by joining the feet of the medians.

**Giornale di Matematiche*, Naples, 1884, July, Vol. XXII, pp. 207–8.

108. $\Delta = \dfrac{L}{2\cos A \cos B \cos C}$, where L is the area of the pedal triangle or triangle formed by joining the feet of the perpendiculars.

109. $\Delta = M\dfrac{(a+b)(b+c)(c+a)}{2abc}$, where M is the area of the triangle formed by joining the feet of the internal bisectors.

110. $\Delta = \dfrac{2R}{r} \cdot N$, where N is the area of the triangle formed by joining the points of tangency of the in-circle.

CONCLUDING NOTE.—Owing to some changes and additions in the foregoing article made since its original preparation and while in press, the statement on page 135, Vol. I, that "the total number of formulæ * * * in this collection is *ninety-three*," etc., is no longer true.

The number of formulæ in the collection is shown below where the second column shows the number that result from counting as distinct those formulæ arising from permutation.

Group I,	31	31
Group II,	41	123
Group III,	11	44
Group IV,	10	60
Group V,	1	12
Miscellaneous Group,	16	18
Total,	110	288

[In the original manuscript all the forms arising from permuting the letters were given in full. To save space in printing, those forms which arose from mere cyclical permutation were omitted.—*O. S.*]

PROBLEM AND SOLUTION.

By DR. J. E. HENDRICKS, Des Moines, Iowa.

PROBLEM.—To inscribe a rectangle whose sides are c,x in another rectangle whose sides are a,b, and express x in functions of a, b, and c.

SOLUTION.—Let $ABCD$ represent the rectangle a,b, and $MNPQ$ the rectangle c,x, and put angle ANM = angle $BPN = \varphi$. Then is

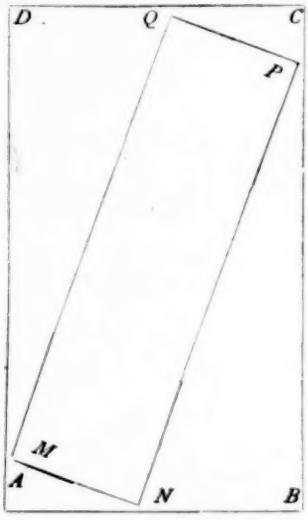
$$x \cos \varphi + c \sin \varphi = a, \tag{1}$$

$$x \sin \varphi + c \cos \varphi = b. \tag{2}$$

From equations (1) and (2) we readily find

$$\cos \varphi = \frac{ax - bc}{x^2 - c^2}$$

$$\sin \varphi = \frac{bx - ac}{x^2 - c^2}$$



Squaring, adding, and reducing, we get

$$x^4 - (a^2 + b^2 + 2c^2)\, x^2 + 4abcx - c^2 (a^2 + b^2 - c^2) = 0. \tag{3}$$

Putting, for brevity,

$$a^2 + b^2 + 2c^2 = m,$$

$$4abc = n,$$

and $$(a^2 + b^2 - c^2)\, c^2 = r,$$

(3) becomes $$x^4 - mx^2 + nx - r = 0. \tag{3'}$$

Transposing (3′) we have,

$$x^4 = mx^2 - nx + r.$$

Adding $2kx^2 + k^2$ to both sides of this equation it becomes

$$x^4 + 2kx^2 + k^2 = (m + 2k)\, x^2 - nx + (k^2 + r),$$

or $$x^2 + k = \sqrt{[(m + 2k)\, x^2 - nx + (k^2 + r)]}. \tag{4}$$

Now, in order that the expression under the radical in (4) shall be an exact square, we must have

$$2\sqrt{(m + 2k)} \times \sqrt{(k^2 + r)} = n,$$

or $$(m + 2k) \times (k^2 + r) = \tfrac{1}{4} n^2.$$

Performing the multiplication indicated and transposing, we get

$$k^3 + \tfrac{1}{2} mk^2 + rk - (\tfrac{1}{8} n^2 - \tfrac{1}{2} mr) = 0. \tag{5}$$

Equation (5) is a complete cubic involving k as the unknown quantity. In

order to apply Cardan's formula to the solution of (5) we must first transform (5) by substituting $z - \frac{1}{6}m$ for k, by which it becomes

$$z^3 - (\tfrac{1}{12}m^2 - r)\,z - \tfrac{1}{2}n^2 + \tfrac{1}{108}m^3 + \tfrac{1}{3}mr = 0. \qquad (6)$$

Putting $p = (\frac{1}{12}m^2 - r),\ q = -\frac{1}{8}n^2 + \frac{1}{108}m^3 + \frac{1}{3}mr$

(6) becomes $z^3 - pz + q = 0. \qquad (6')$

By applying Cardan's formula to (6′) we get

$$z = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}}. \qquad (7)$$

If we assume $a = 3$, $b = 2$, and $c = 1$, we find, by substitution, $p = 6\frac{3}{4}$ and $q = 19\frac{1}{4}$, and substituting these values in (7) we find

$$z = -3\tfrac{1}{2}.$$

And because $k = z - \frac{1}{6}m$,

$$\therefore\ k = -3\tfrac{1}{4} - 2\tfrac{1}{2} = -6.$$

Substituting this value of k in (4) the equation becomes

$$x^2 - 6 = -\sqrt{3}\,.\,x + \sqrt{48}, \qquad (8)$$

whence, transposing, completing the square and extracting the roots, we find

$$x = 2.8323801.$$

The roots of (3′) may be constructed by intersecting two plane curves, as follows: —

If we put $m' = \frac{1}{4}(m + 4),\ \ n' = -\frac{1}{8}n$

and $r' = \frac{1}{8}\sqrt{4m^2 + n^2 + 32m + 16r + 64},$

and construct a parabola whose parameter is 2; then, regarding the vertex of the parabola as the origin of co-ordinates, if we describe a circle, with radius r and co-ordinates of centre m', n', the perpendicular distances from the axis of the parabola to the intersections of the peripheries of the two curves will represent the roots of the equation.

Whatever values may be assigned to a, b, and c, as applied in this problem, the solution of (3′) will always give two imaginary roots, and two real, one positive and one negative. [*J. E. Hendricks.*]

SOLUTIONS OF EXERCISES.

27

THE NUMBER of points common to three surfaces of the mth, nth, and pth degrees being in general mnp, find the co-ordinates of all the real and imaginary points of intersection of the three surfaces, $y^2 + x^2 + z^2 = 4x$, $x^2 = yz$, and $y^2 = x^4$. [*H. A. Newton.*]

SOLUTION.

The three surfaces $x^2 + y^2 + z^2 - 4x$, $x^2 - yz$, $x^4 - y^2$ are respectively a sphere, a hyperbolic parabaloid, and a pair of parabolic cylinders $x^2 \pm y$. The parabolic cylinders intersect the hyperbolic paraboloid in the planes $z \pm 1$ and at infinity. These planes cut the sphere in the circles $x^2 + y^2 - 4x + 1$ which intersect the parabolas $x^2 \pm y$ in the points whose abscissæ are the roots of

$$x^4 + x^2 - 4x + 1 = 0;$$

that is $x_1 = 0.27$; $x_2 = 1.25$; $x_3 = -0.76 + 1.54i$; $x_4 = -0.76 - 1.54i$.

The corresponding values of y are

$$y_1 = \pm 0.07; \; y_2 = \pm 1.56; \; y_3 = \mp (1.79 + 2.34i); \; y_4 = \mp (1.79 - 2.34i).$$

In addition to these eight intersection points, four real and four imaginary, there are altogether at infinity eight other intersection points, which make up the full number of sixteen ($2 \times 2 \times 4 = 16$). [*Wm. M. Thornton.*]

46

FIND an equation that shall express approximately the surface of an egg shell. By means of the variations of not more than three constants in the equations, express the various sizes and shapes of eggs. [*H. A. Newton.*]

SOLUTION.

Assume a fixed circle of radius r and centre C, and a fixed point O. Join O to any point P on the circumference and on PO lay off $PM = c$. If then

$$OM = \rho, \quad COM = \theta, \quad OC = a,$$

it easy to show by projecting OC on OP that

$$\rho = a \cos\theta - c - \sqrt{(r^2 - a^2 \sin^2\theta)}.$$

This equation represents the locus of M referred to O as pole. This locus is an

ovoid curve whose rotation about OC generates the egg shell. By varying the constants a,c,r we obtain the various sizes and shapes of eggs. In particular, if $c = 0$ the curve is the base circle and the egg is spherical.

[*J. E. Hendricks.*]

47

Two bulls arriving on opposite banks of a river a rods wide, plunge into the water at the same instant and swim toward each other until they meet. One bull can swim m miles and the other n ($m > n$) miles an hour in still water, and the velocity of the river is v miles an hour. Required the equation to the curve described and the distance swum by each bull and the location of the point of meeting. [*Artemas Martin.*]

SOLUTION.

Let A,B be the bulls; P,Q their positions at any instant; and P_0,Q_0 their initial positions. Then since the velocity of the stream is everywhere constant, PQ will remain always parallel to P_0Q_0; and the bulls will meet after t seconds where

$$mt + nt = a \operatorname{cosec} \beta,$$

a being the breadth of the stream and β the angle between P_0Q_0 and the bank.

Since the component velocities of A,B are constant the paths will be straight. The resultant velocities will be

$$u = \sqrt{(m^2 + v^2 - 2mv \cos \beta)},$$
$$w = \sqrt{(n^2 + v^2 + 2nv \cos \beta)},$$

and the spaces described will be ut, wt.

[*J. E. Hendricks.*]

48

In the angle A a point P is given. Construct a triangle ABC whose base BC shall contain P while the sum of the sides AB and AC equals a given length.

[*W. M. Thornton.*]

SOLUTION.

It is readily shown that the line

$$\frac{x}{a} + \frac{y}{k - a} = 1$$

is tangent to the parabola

$$x^2 - 2xy + y^2 - 2k(x + y) + k^2 = 0.$$

Setting $AB + AC = K$, a knowledge of the fundamental properties of tangents to a parabola makes the following construction self-explanatory:—

Lay out on AB, AC, respectively, AX, AY, each equal to K. Draw AN through N, the middle point of XY. V, the middle point of AN, is the principal vertex of the parabola mentioned above. Draw VZ perpendicular to AV, cutting AX in Z; then ZS perpendicular to AX, cutting AN in S, the focus of the parabola. Over SP as a diameter draw a circle cutting VZ in Q,R. Draw PQ, PR; these are tangents to the parabola and cut AX, AY in B,C, giving two constructions. [*R. D. Bohannan.*]

EXERCISES.

51

A QUADRILATERAL, both circumscriptible and inscriptible, has a given perimeter $4s$ and one given angle α. Show that its area is

$$2s^2 : (\operatorname{cosec} \alpha + \operatorname{cosec} x)$$

where x is one of the angles adjacent to α. And find x when the area is greatest. [*W. M. Thornton.*]

52

FIND the height to which the Washington Monument must be built so that a body placed on top of it would have no weight. [*A. Hall.*]

53

SHOW that when $x = \frac{2\pi}{13}$

$$(\cos x + \cos 5x)(\cos 2x + \cos 3x)(\cos 4x + \cos 6x) = -\tfrac{1}{8}.$$

[*R. H. Graves.*]

54

SHOW that if in a plane triangle ABC

$$\cos A + \cos B + \cos C = 1\sqrt{2},$$

the centre of the circumscribed circle lies on the circumference of the inscribed circle. [*R. D. Bohannan.*]

55

WHAT is the locus of the vertex of a variable angle whose sides intercept a fixed line and whose bisectrix passes through a given point in that line.

[*L. G. Carpenter.*]

SELECTED.—CONSTRUCTIVE PROBLEMS IN ELEMENTARY GEOMETRY.

56

CONSTRUCT a triangle; given its circumscribed circle and the radii which bisect its sides.

57

A TRANSVERSAL through the point O intersects the lines a,b,c in the points A,B,C; construct the transversal on which OA equals BC.

58

DIVIDE an isosceles triangle by perpendiculars from a point within it to the sides, into three equivalent quadrilaterals.

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59

SHOW that the height of a cone bears to the height H of an equal spherical dome of diameter D on the same base the ratio

$$3D - 2H : 2D - 2H.$$

60

DIVIDE a sphere by a secant plane into two segments whose entire surfaces shall be as 1 to 3.

61

DIVIDE a sphere by a secant plane into two parts whose volumes shall be as 1 to 3.

62

FIND the radius of a spherical dome whose volume is 2000^{cc} and whose altitude is $\frac{2}{3}$ of the radius.

63

FIND the radius of a spherical dome whose surface shall be 2000^{qc} and whose altitude is $\frac{2}{3}$ of the radius.

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